Stringed Musical Instruments

Prof. James L. Buchanan
Mathematics Department

November 17, 2004

We have shown that the solution to the plucked string problem

$$\begin{aligned}
\frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial x^2} \\
v(0,t) &= 0, v(L,t) = 0 \\
v(x,0) &= f(x), \frac{\partial v}{\partial t}(x,0) = 0
\end{aligned}$$

is

$$v(x,t) = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}, C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{1}$$

where $c^2 = T_0/\rho_0$ is the ratio of tension to linear density. Thus the motion of a string is the superposition of infinitely many *modes of vibration* $\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$. The frequency of the n^{th} mode is $f_n = \frac{nc}{2L}$. The amplitudes C_n of the modes typically decrease with increasing n and thus for a vibrating string the loudest frequency is usually $f_1 = \frac{c}{2L}$. This is the *fundamental tone, first harmonic,* or *pitch* of the string. Clearly any pitch is attainable by using the appropriate combination of length L, tension T_0 and density ρ_0. The higher frequency modes are referred to as *overtones* or *higher harmonics.*

If we take the initial shape of the string to be that which would arise from plucking it to a height a at the point $x = \alpha$ then

$$f(x) = \begin{cases} \frac{a}{\alpha}x & 0 \le x \le \alpha \\ \frac{a}{L-\alpha}(L-x) & \alpha < x \le L \end{cases}.$$

The coefficients in (1) are then

$$C_n = \frac{2aL^2}{\alpha(L-\alpha)n^2\pi^2}\sin\left(\frac{n\pi\alpha}{L}\right).$$

Figures 1 and 2 indicate the shapes and amplitudes of the first five modes $C_n \sin \frac{n\pi x}{L}$ for a string of length $L = 1$ which is plucked to a height $a = 1$ at $\alpha = 0.5$ and $\alpha = 0.1$ respectively.

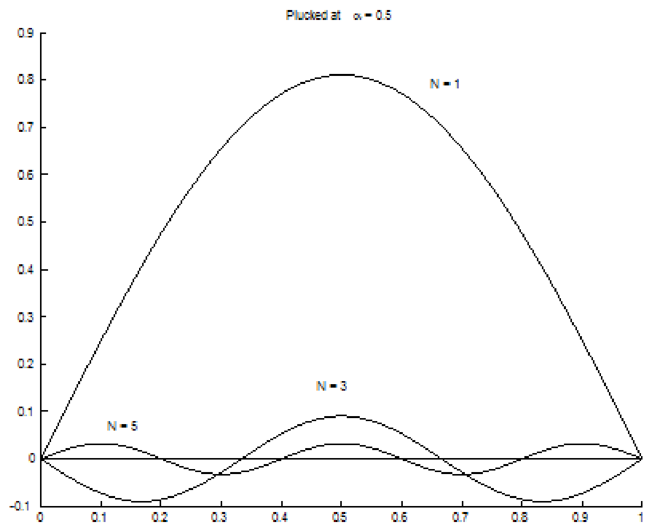

Figure 1: Amplitudes of the first five modes for a string plucked in the middle.

It has been known since the time of Pythagoras that two strings of similar density held at the same tension and plucked simultaneously will produce a sound pleasing to the ear, providing that the lengths of the strings are in the ratio $m : n$ for small integers m and n. This also explains why a single plucked string makes a pleasing sound: the frequencies of the modes satisfy $f_n = nf_1$ and thus the frequencies of the loudest modes have small integer ratios to each other.

Some musical terminology: *Intervals* are the ideal ratios between the frequencies of notes. For example C to G is a perfect fifth when the ratio of the two frequencies is 3:2. The Table below gives the intervals commonly used in music and their ideal ratios. The listing is chronological: The first four intervals (*perfect consonances*) were used as the basis of harmony in the eleventh through fifteenth centuries, the second four (*imperfect consonances*) in the sixteenth through nineteenth centuries. The last four (*dissonances*)

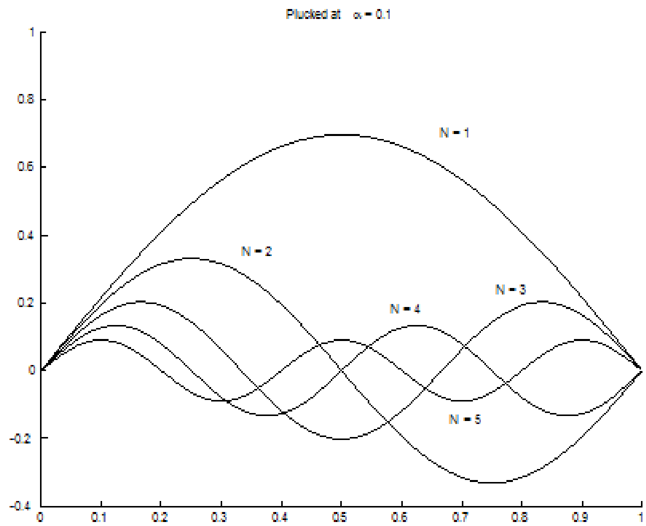

Figure 2: Amplitudes of the first five modes for a string plucked of to the side.

were not much used until the twentieth century.

Interval	**Ratio**
Unison	1:1
Octave	2:1
Perfect fifth	3:2
Perfect fourth	4:3
Major third	5:4
Minor third	6:5
Major sixth	5:3
Minor sixth	8:5
Minor seventh	9:5
Major seventh	15:8
Minor second	10:9
Major second	9:8

For an instrument such as a piano which covers several octaves these ideal ratios are not attainable. For instance the note A above middle C that an orchestra tunes to before a concert is currently 440 Hz[1]. If we tuned a piano

[1] The frequency of this note has varied over the centuries. In the seventeenth century

so that the interval between the concert A and the note D above it was a perfect fourth, then its frequency would be $4/3 \times 440 = 586.67$ Hz. If we make the interval from this D to the G above it a perfect fourth, then the frequency for this G is $(4/3)^2 \times 440 = 782.22$ Hz. If we continued in this manner then the A three octaves above the original one would have frequency $(4/3)^7 \times 440 = 3296.3$ Hz. On the other hand going up in units of an octave gives $2^3 \times 440 = 3520.0$ Hz for the same A and thus there is more than a 200 Hz discrepancy. To resolve this problem pianos and other keyboard instruments use a *tempered scale* with twelve semitones per octave. Thus for an A with a frequency of 440 Hz $A\sharp$ (= B$\flat$) is $440 \times 2^{1/12} = 466.2$, B is $440 \times 2^{1/6} = 493.9$ and so forth. To get to twelve semitones from the fourteen that there should be, the identifications $B\sharp = C$ and $E\sharp = F$ are made. Thus a perfect fourth becomes an imperfect fourth in which the frequency is multiplied by $2^{5/12} = 1.3348$ rather than $4/3$.

it had several pitches ranging from 373.7 to 402.9 Hz. By the middle of the eighteenth century the concert A was 422.9, but for other purposes A's as high as 461 were used. In 1834 a group of German physicists advocated the use of the modern value of 440, but the value of 435 adopted by France in 1859 became standard. This value was used until recently when the United States adopted 440 Hz.